STEP MATHEMATICS 3

2020

Worked Solutions

STEP 3: BRIEF SOLUTIONS

1. (i) Integrating by parts,

$u = \cos^a x \qquad v' = \cos bx$
$u' = -a\cos^{a-1}x\ \sin x \qquad v = \frac{1}{b}\sin bx$

$$I(a,b) = \left[\cos^a x\ \frac{1}{b}\sin bx\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -a\cos^{a-1}x\ \sin x\ \frac{1}{b}\sin bx\ dx$$

$$= 0 + \int_0^{\frac{\pi}{2}} a\cos^{a-1}x\ \sin x\ \frac{1}{b}\sin bx\ dx = \frac{a}{b}\int_0^{\frac{\pi}{2}} \cos^{a-1}x\ \sin x\ \sin bx\ dx$$

$\cos(b-1)x = \cos bx\cos x + \sin bx\sin x$

So $I(a,b) = \frac{a}{b}\ \int_0^{\frac{\pi}{2}} \cos^{a-1}x\ (\cos(b-1)x - \cos bx\cos x)\ dx$
$= \frac{a}{b}[I(a-1,b-1) - I(a,b)]$

Thus $I(a,b) = \frac{a}{a+b}\ I(a-1,b-1)$ as required.

(ii) Suppose

$$I(k,k+2m+1) = (-1)^m\frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!}$$

Then by (i),

$$I(k+1,k+2m+2) = \frac{k+1}{2k+2m+3}I(k,k+2m+1)$$

$$= \frac{k+1}{2k+2m+3}\ (-1)^m\frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!}$$

$$= \frac{k+1}{2k+2m+3}\ (-1)^m\frac{2^k\ k!\ (2m)!\ (k+m)!}{m!\ (2k+2m+1)!}\times\frac{2k+2m+2}{2k+2m+2}$$

$$= (-1)^m\ \frac{2^{k+1}\ (k+1)!\ (2m)!\ (k+m+1)!}{m!\ (2k+2m+3)!}$$

$$= (-1)^m\ \frac{2^{k+1}\ (k+1)!\ (2m)!\ \big((k+1)+m\big)!}{m!\ (2(k+1)+2m+1)!}$$

which is the required result for $k+1$

$$I(0,2m+1) = \int_0^{\frac{\pi}{2}} \cos(2m+1)x \ dx = \frac{1}{2m+1}\left[\sin(2m+1)x\right]_0^{\frac{\pi}{2}}$$

$= \frac{1}{2m+1}$ if m is even or $= \frac{-1}{2m+1}$ if m is odd , or alternatively $(-1)^m \frac{1}{2m+1}$

If $n = 0$,

$$(-1)^m \frac{2^n \ n! \ (2m)! \ (n+m)!}{m! \ (2n+2m+1)!} = (-1)^m \frac{(2m)! \ (m)!}{m! \ (2m+1)!} = (-1)^m \frac{1}{2m+1}$$

so result is true for $n = 0$.

So by the principle of mathematical induction, the required result is true.

Alternative for (i)

$$\cos^a x \cos bx = \cos^{a-1} x \left[\cos x \cos bx\right]$$
$$\cos x \cos bx = \frac{1}{2}\left[\cos(b+1)x + \cos(b-1)x\right]$$
$$2I(a,b) = I(a-1,b+1) + I(a-1,b-1)$$

Also

$$\sin x \sin bx = \frac{1}{2}\left[\cos(b-1)x - \cos(b+1)x\right]$$

so using integration by parts of main scheme,

$$2I(a,b) = \frac{a}{b}\left[I(a-1,b-1) - I(a-1,b+1)\right]$$

Eliminating $I(a-1,b+1)$ between these results gives required result.

2. (i) $\sinh x + \sinh y = 2k$

Differentiating with respect to x, $\cosh x + \cosh y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0 \Rightarrow \cosh x = 0$ which is not possible as $\cosh x \geq 1 \ \forall x$, so there are no stationary points.

Differentiating again with respect to x, $\sinh x + \sinh y \left(\frac{dy}{dx}\right)^2 + \cosh y \frac{d^2y}{dx^2} = 0$

$\frac{dy}{dx} = \frac{-\cosh x}{\cosh y}$ and $\frac{d^2y}{dx^2} = 0$ implies $\sinh x + \sinh y \left(\frac{-\cosh x}{\cosh y}\right)^2 = 0$

$$\cosh^2 y \sinh x + \cosh^2 x \sinh y = 0$$

$$(1 + \sinh^2 y) \sinh x + (1 + \sinh^2 x) \sinh y = 0$$

$$(\sinh x + \sinh y)(1 + \sinh x \sinh y) = 0$$

But $\sinh x + \sinh y = 2k > 0$ so

$$1 + \sinh x \sinh y = 0$$

as required.

At a point of inflection, $\frac{d^2y}{dx^2} = 0$, so $\sinh x + \sinh y = 2k$ and $\sinh x \sinh y = -1$ and thus, $\sinh x$ (and $\sinh y$ as well) is a root of $\lambda^2 - 2k\lambda - 1 = 0$

$$\lambda = \frac{2k \pm \sqrt{4k^2 + 4}}{2}$$

$\sinh x = k + \sqrt{k^2 + 1}$, $\sinh y = \frac{-1}{k+\sqrt{k^2+1}} = \frac{-1}{k+\sqrt{k^2+1}} \times \frac{k-\sqrt{k^2+1}}{k-\sqrt{k^2+1}} = \frac{-(k-\sqrt{k^2+1})}{k^2-(k^2+1)} = k - \sqrt{k^2 + 1}$

and vice versa.

So the points of inflection are

$\left(\sinh^{-1}\left(k + \sqrt{k^2 + 1}\right), \ \sinh^{-1}\left(k - \sqrt{k^2 + 1}\right)\right)$ and

$\left(\sinh^{-1}\left(k - \sqrt{k^2 + 1}\right), \ \sinh^{-1}\left(k + \sqrt{k^2 + 1}\right)\right)$

(ii) $x + y = a \Rightarrow y = a - x$ so as $\sinh x + \sinh y = 2k$

$$\frac{e^x - e^{-x}}{2} + \frac{e^{a-x} - e^{x-a}}{2} = 2k$$

Multiplying by $2e^x$,

$$e^{2x} - 1 + e^a - e^{2x}e^{-a} = 4ke^x$$

$$e^{2x}(1 - e^{-a}) - 4ke^x + (e^a - 1) = 0$$

As e^x is real, $'b^2 - 4ac \geq 0'$, so $16k^2 - 4(1 - e^{-a})(e^a - 1) \geq 0$

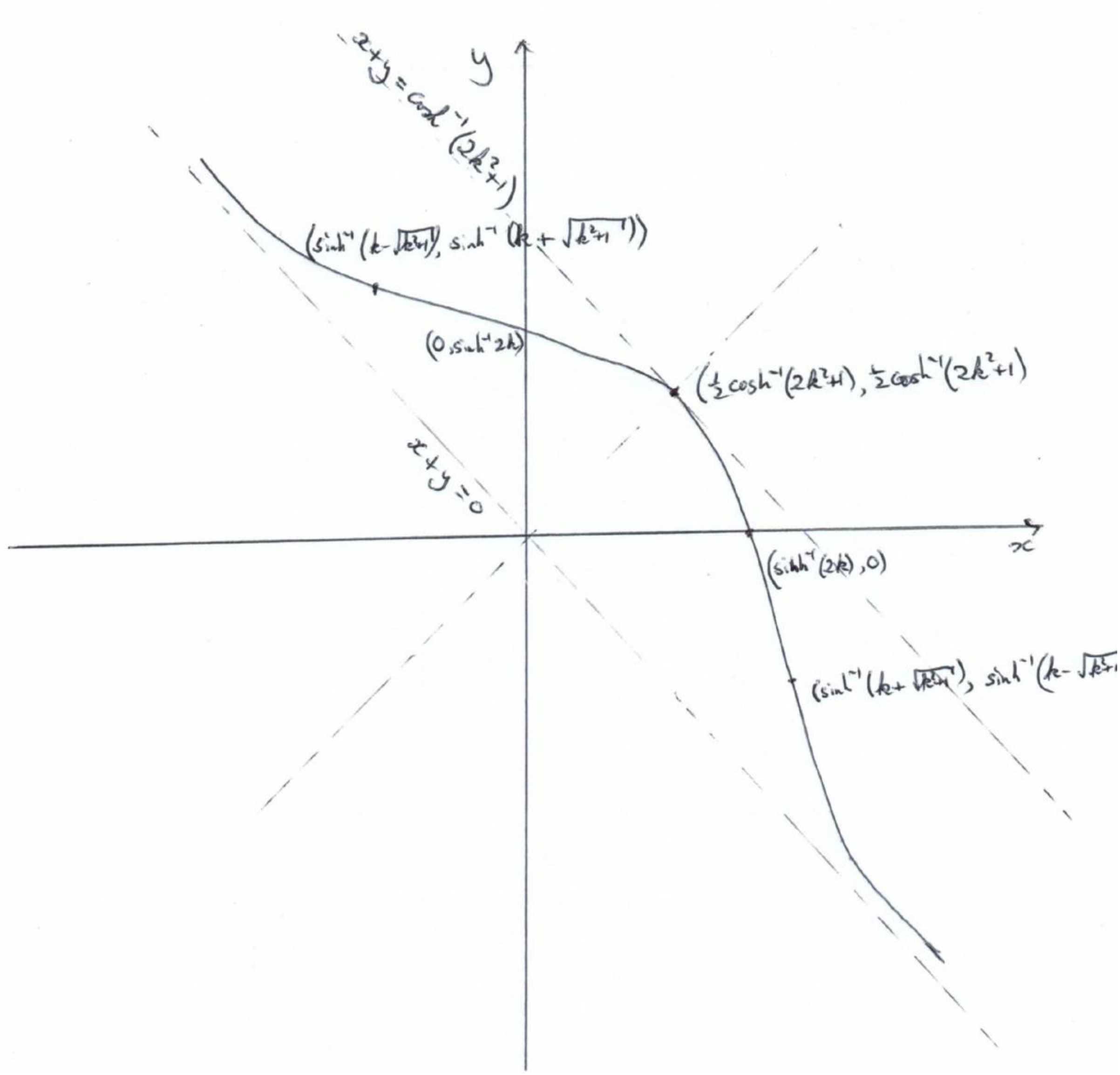
$$4k^2 - e^a - e^{-a} + 2 \geq 0$$
$$4k^2 - 2\cosh a + 2 \geq 0$$

So $\cosh a \leq 2k^2 + 1$

If $a = 0$, then $x = -y$ so $\sinh x = -\sinh y$ and thus $\sinh x + \sinh y = 2k = 0$ but $k > 0$.

So $\cosh a > 1$ as required.

(iii)



Alternative

(i) $\frac{dy}{dx} = \frac{-\cosh x}{\cosh y}$, $\frac{d^2y}{dx^2} = -\left\{\frac{\cosh y \sinh x - \cosh x \sinh y \frac{dy}{dx}}{\cosh^2 y}\right\} = -\left\{\frac{\cosh^2 y \sinh x + \cosh^2 x \sinh y}{\cosh^3 y}\right\}$

then as before.

(ii) Substituting $a = 0$ would imply $e^x = 0$ which is impossible.

3. (i)

$$k - a = (b - a)e^{-\frac{i\pi}{3}}$$

Therefore,

$$g_{AB} = \frac{1}{3}\left[a + b + \left(a + (b - a)e^{-\frac{i\pi}{3}}\right)\right]$$

$$= a\left(\frac{2 - e^{-\frac{i\pi}{3}}}{3}\right) + b\left(\frac{1 + e^{-\frac{i\pi}{3}}}{3}\right)$$

$$\omega = e^{\frac{i\pi}{6}} = \frac{\sqrt{3} + i}{2}$$

and so

$$\omega^* = \frac{\sqrt{3} - i}{2}$$

$$\frac{2 - e^{-\frac{i\pi}{3}}}{3} = \frac{2 - \left(\frac{1 - i\sqrt{3}}{2}\right)}{3} = \frac{3 + i\sqrt{3}}{6} = \frac{1}{\sqrt{3}}\frac{\sqrt{3} + i}{2} = \frac{1}{\sqrt{3}}\omega$$

and

$$\frac{1 + e^{-\frac{i\pi}{3}}}{3} = \frac{1 + \left(\frac{1 - i\sqrt{3}}{2}\right)}{3} = \frac{3 - i\sqrt{3}}{6} = \frac{1}{\sqrt{3}}\omega^*$$

Thus $g_{AB} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b)$ as required.

(ii) $g_{AB} = \frac{1}{\sqrt{3}}(\omega a + \omega^* b)$

$g_{BC} = \frac{1}{\sqrt{3}}(\omega b + \omega^* c)$

$g_{CD} = \frac{1}{\sqrt{3}}(\omega c + \omega^* d)$

$g_{DA} = \frac{1}{\sqrt{3}}(\omega d + \omega^* a)$

Q_1 parallelogram $\Rightarrow b - a = c - d \Leftrightarrow d - a = c - b$

$$g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b - a) + \omega^*(c - b)\big) = \frac{1}{\sqrt{3}}\big(\omega(c - d) + \omega^*(d - a)\big) = g_{CD} - g_{DA}$$

$\Rightarrow Q_2$ parallelogram.

Q_2 parallelogram $\Rightarrow g_{BC} - g_{AB} = g_{CD} - g_{DA}$

$$\frac{1}{\sqrt{3}}\{\omega[(b - a) - (c - d)] + \omega^*[(c - b) - (d - a)]\} = 0$$

$$\frac{1}{\sqrt{3}}(\omega^* - \omega)[(a - b) - (d - c)] = 0$$

As $\omega^* - \omega \neq 0$, $(a - b) - (d - c) = 0$ and so Q_1 is a parallelogram

(iii)

$$g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b-a) + \omega^*(c-b)\big)$$

$$g_{CA} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(c-a) + \omega^*(a-b)\big) \quad (1)$$

$$\omega^2(g_{BC} - g_{AB}) = \frac{1}{\sqrt{3}}\big(\omega^3(b-a) + \omega(c-b)\big) \quad (2)$$

$$= \frac{1}{\sqrt{3}}\big(i(b-a) + \omega(c-b)\big)$$

The coefficient of $\frac{1}{\sqrt{3}}a$ in (1) is $\omega^* - \omega = \frac{\sqrt{3}-i}{2} - \frac{\sqrt{3}+i}{2} = -i$
The coefficient of $\frac{1}{\sqrt{3}}b$ in (1) is $-\omega^* = (i - \omega)$
The coefficient of $\frac{1}{\sqrt{3}}c$ in (1) is ω
Thus $G_{AB}G_{BC}$ rotated through $\frac{\pi}{3}$ is $G_{AB}G_{CA}$ which means that $G_{AB}G_{BC}G_{CA}$ is an equilateral triangle.

(iii) Alternative

$$x = g_{BC} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(b-a) + \omega^*(c-b)\big)$$

$$y = g_{CA} - g_{AB} = \frac{1}{\sqrt{3}}\big(\omega(c-a) + \omega^*(a-b)\big)$$

$$x = \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{6}}b - e^{\frac{i\pi}{6}}a + e^{\frac{-i\pi}{6}}c - e^{\frac{-i\pi}{6}}b\right)$$

$$= \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{2}}b + e^{\frac{-i\pi}{6}}c + e^{\frac{i7\pi}{6}}a\right)$$

$$y = \frac{1}{\sqrt{3}}\left(e^{\frac{i\pi}{6}}c + e^{\frac{i3\pi}{2}}a + e^{\frac{i5\pi}{6}}b\right)$$

$$\frac{y}{x} = e^{\frac{i\pi}{3}}$$

$e^{\frac{i\pi}{3}}$ means y is x rotated through $\frac{\pi}{3}$ and thus ABC is an equilateral triangle.

[or alternatively

$\left|\frac{y}{x}\right| = 1$ and similarly $\left|\frac{z}{y}\right| = 1$ and thus all three sides are equal length]

4. π has equation $r.n = 0$ so n is a vector perpendicular to this plane.

Q lies on π if $x - (x.n)n$ satisfies $r.n = 0$

$(x - (x.n)n).n = x.n - (x.n)n.n = x.n - x.n = 0$ so Q lies on π as required.

$PQ = (x - (x.n)n) - x = -(x.n)n$ which is parallel to n and so is perpendicular to π .

(i) The image of a point with position vector x under T is $x - 2(x.n)n$, so as $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and

$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the image of i under T is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2a\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1-2a^2 \\ -2ab \\ -2ac \end{pmatrix}$

But $a^2 + b^2 + c^2 = 1$ so $1 - 2a^2 = a^2 + b^2 + c^2 - 2a^2 = b^2 + c^2 - a^2$

Thus, the image of i under T is $\begin{pmatrix} b^2 + c^2 - a^2 \\ -2ab \\ -2ac \end{pmatrix}$ as required.

Similarly, the images of j and k are $\begin{pmatrix} -2ab \\ c^2 + a^2 - b^2 \\ -2bc \end{pmatrix}$ and $\begin{pmatrix} -2ac \\ -2bc \\ a^2 + b^2 - c^2 \end{pmatrix}$ respectively.

Thus $M = \begin{pmatrix} b^2 + c^2 - a^2 & -2ab & -2ac \\ -2ab & c^2 + a^2 - b^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix}$

(ii) $1 - 2a^2 = 0.\cdot 64 \Rightarrow a = \pm 0 \cdot 3\sqrt{2}$ and thus as $-2ab = 0 \cdot 48$ and $-2ac = 0 \cdot 6$,

$b = \mp 0 \cdot 4\sqrt{2}$ and $c = \mp 0 \cdot 5\sqrt{2}$ and the plane is $3x - 4y - 5z = 0$ (or $-3x + 4y + 5z = 0$)

(iii) Suppose the position vector of the point Q on the given line such that PQ is perpendicular to that line is y , then $y = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some λ and $(y - x).\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

So, $y.\begin{pmatrix} a \\ b \\ c \end{pmatrix} - x.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$, i.e. $\lambda = x.\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

So, the image of P under the rotation, is $x + 2(y - x) = 2y - x = 2x.\begin{pmatrix} a \\ b \\ c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} - x$

The image of i under the rotation is thus $\begin{pmatrix} 2a^2 - 1 \\ 2ab \\ 2ac \end{pmatrix} = \begin{pmatrix} a^2 - b^2 - c^2 \\ 2ab \\ 2ac \end{pmatrix}$, and of j and k are $\begin{pmatrix} 2ab \\ b^2 - c^2 - a^2 \\ 2bc \end{pmatrix}$ and $\begin{pmatrix} 2ac \\ 2bc \\ c^2 - a^2 - b^2 \end{pmatrix}$ respectively.

Thus $N = \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-c^2-a^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix}$, which, incidentally $= -M$.

(iv) $NM = -MM = -I$ as M is self-inverse.

Thus the single transformation is an enlargement, scale factor -1 , with centre of enlargement the origin.

alternative for (iii)

$x = u + v$ where $u \in \Pi$ and $v \perp \Pi$

$Mv = -v \qquad Mu = u \qquad Mx = u - v$

$Nx = v - u = -Mx$

$N = -M$

alternative for (ii) the matrix represents a reflection, an invariant point under the reflection lies on the plane of reflection.
Therefore,

$$\begin{pmatrix} 0.64 & 0.48 & 0.6 \\ 0.48 & 0.36 & -0.8 \\ 0.6 & -0.8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Taking the simplest component equation

$$0.6x - 0.8y = z$$

(although the other two give equivalent equations).

This simplifies to
$3x - 4y - 5z = 0$

5. $(x-y)(x^{n-1}+x^{n-2}y+\cdots+y^{n-1}) = x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y + \cdots + xy^{n-1} - y^n$

$$= x^n - y^n$$

as each even numbered term cancels with its subsequent term.

(i) If

$$F(x) = \frac{1}{x^n(x-k)} = \frac{A}{x-k} + \frac{f(x)}{x^n}$$

then multiplying by $x^n(x-k)$

$$1 = Ax^n + (x-k)f(x)$$

$$x = k \Rightarrow A = \frac{1}{k^n}$$

so

$$1 = \frac{x^n}{k^n} + (x-k)f(x)$$

and

$$f(x) = \frac{1}{x-k}\left(1 - \left(\frac{x}{k}\right)^n\right)$$

as required.

Thus

$$F(x) = \frac{\frac{1}{k^n}}{x-k} + \frac{\frac{1}{x-k}\left(1-\left(\frac{x}{k}\right)^n\right)}{x^n}$$

$$= \frac{1}{k^n(x-k)} - \frac{x^n - k^n}{k^n x^n (x-k)}$$

and so, by the result of the stem,

$$F(x) = \frac{1}{k^n(x-k)} - \frac{1}{k^n x^n}\sum_{r=1}^{n} x^{n-r}k^{r-1}$$

$$= \frac{1}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{1}{k^{n-r}x^r}$$

(ii)

$$x^nF(x) = \frac{1}{x-k} = \frac{x^n}{k^n(x-k)} - \frac{1}{k}\sum_{r=1}^{n}\frac{x^{n-r}}{k^{n-r}}$$

Differentiating with respect to x ,

$$\frac{-1}{(x-k)^2} = \frac{nx^{n-1}}{k^n(x-k)} - \frac{x^n}{k^n(x-k)^2} - \frac{1}{k}\sum_{r=1}^{n}\frac{(n-r)x^{n-r-1}}{k^{n-r}}$$

Multiplying by $\frac{-1}{x^n}$

$$\frac{1}{x^n(x-k)^2} = \frac{-n}{xk^n(x-k)} + \frac{1}{k^n(x-k)^2} + \sum_{r=1}^{n}\frac{n-r}{k^{n+1-r}x^{r+1}}$$

(iii)

$$\int_2^N \frac{1}{x^3(x-1)^2}\,dx = \int_2^N \frac{-3}{x(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\ dx$$

$$= \int_2^N \frac{3}{x} - \frac{3}{(x-1)} + \frac{1}{(x-1)^2} + \sum_{r=1}^{3}\frac{3-r}{x^{r+1}}\ dx$$

$$= \left[3\ln x - 3\ln(x-1) - \frac{1}{x-1} - \sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$= \left[3\ln\left(\frac{x}{x-1}\right) - \frac{1}{x-1} - \sum_{r=1}^{3}\frac{3-r}{rx^r}\right]_2^N$$

$$= 3\ln\left(\frac{N}{N-1}\right) - \frac{1}{N-1} - \sum_{r=1}^{3}\frac{3-r}{rN^r} - 3\ln(2) + 1 + \sum_{r=1}^{3}\frac{3-r}{r2^r}$$

As $N \to \infty$, $\left(\frac{N-1}{N}\right) \to 1$, so $3\ln\left(\frac{N-1}{N}\right) \to 0$, and $\frac{1}{N-1} \to 0$, $\frac{1}{N^r} \to 0$

So the limit of the integral is,

$$-3\ln 2 + 1 + \frac{2}{2} + \frac{1}{8} = -3\ln 2 + \frac{17}{8}$$

Alternatives for stem using sum of GP or proof by induction

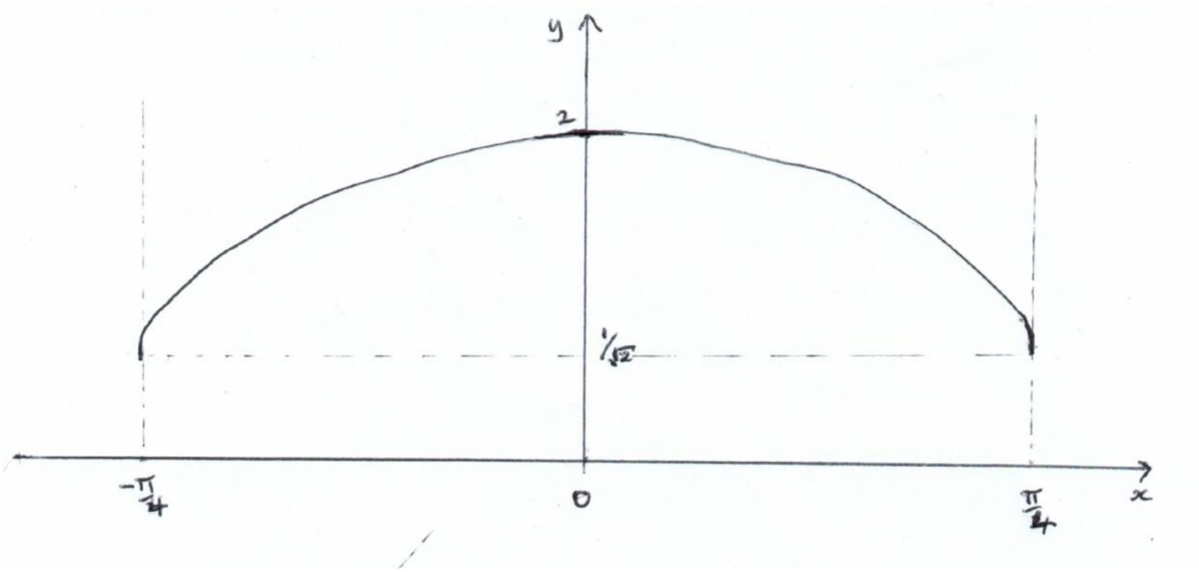
6. (i)

$$y = \cos x + \sqrt{\cos 2x}$$

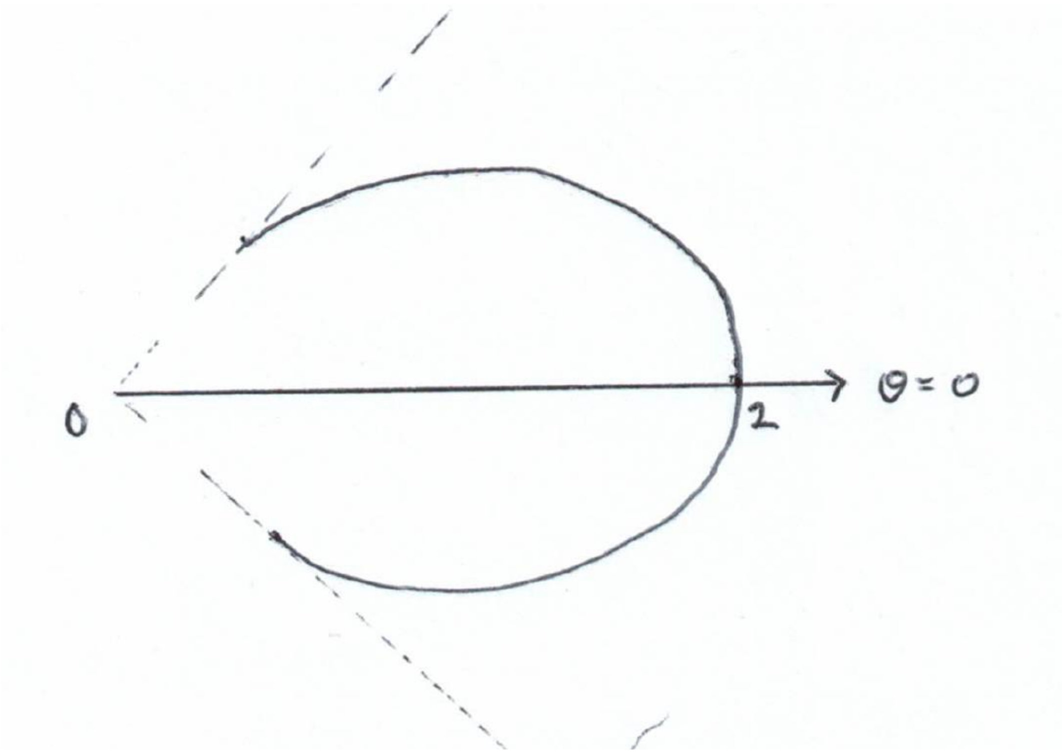
$x = 0\,, y = 2$ There is symmetry in $x = 0$ $x = \pm\frac{\pi}{4}\,, y = \frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = -\sin x - \frac{\sin 2x}{\sqrt{\cos 2x}}$ so $x = 0\,, \frac{dy}{dx} = 0$ $x > 0\,, \frac{dy}{dx} < 0$ and vice versa

as $x \to \frac{\pi}{4}\,, \frac{dy}{dx} \to -\infty$



(ii)



(iii) $\theta = \pm\frac{\pi}{4}$, $r = \frac{1}{\sqrt{2}}$

$r^2 - 2r\cos\theta + \sin^2\theta = 0$

$$(r - \cos\theta)^2 = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

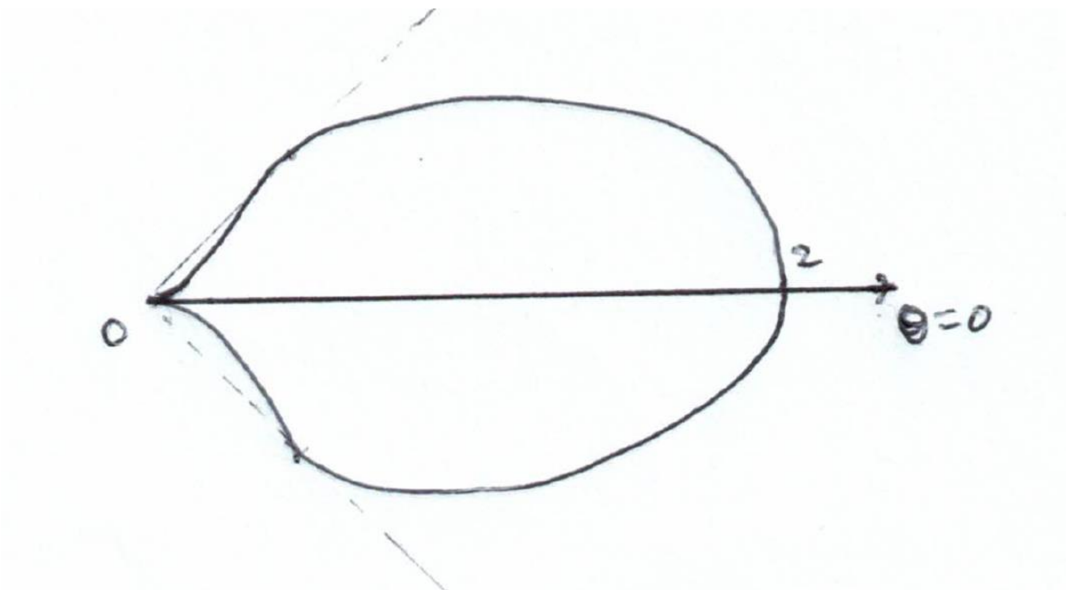
Therefore, $r - \cos\theta = \pm\sqrt{\cos 2\theta}$, i.e. $r = \cos\theta \pm \sqrt{\cos 2\theta}$

From (i), r is only small on the branch, $r = \cos\theta - \sqrt{\cos 2\theta}$. For $\theta = 0$, $r = 0$

Otherwise, $\cos\theta - \sqrt{\cos 2\theta} = 0$, $\cos 2\theta = \cos^2\theta$, $2\cos^2\theta - 1 = \cos^2\theta$, $\cos\theta = \pm 1$ so for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, $\theta = 0$ is the only value for which $r = 0$

So r small implies $r = \cos\theta - \sqrt{\cos 2\theta}$ and θ is small

Thus $r \approx 1 - \frac{\theta^2}{2} - \left(1 - \frac{(2\theta)^2}{2}\right)^{\frac{1}{2}} \approx 1 - \frac{\theta^2}{2} - 1 + \theta^2 = \frac{\theta^2}{2}$ as required.



Area required is

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos\theta + \sqrt{\cos 2\theta}\right)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos\theta - \sqrt{\cos 2\theta}\right)^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{\cos 2\theta}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \cos\theta\ \sqrt{1 - 2\sin^2\theta}\ d\theta$$

Let $\sqrt{2}\sin\theta = \sin u$, then $\sqrt{2}\cos\theta\ \frac{d\theta}{du} = \cos u$, So the integral becomes

$$2\int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{\sqrt{2}}\ du = \sqrt{2}\ \int_0^{\frac{\pi}{2}} \frac{\cos 2u + 1}{2}\ du = \sqrt{2}\left[\frac{\sin 2u}{4} + \frac{u}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{2}}$$

(iii) alternative

$$r \ll 1 \ \Rightarrow\ -2r\cos\theta + \sin^2\theta \approx 0$$

$$r \approx \frac{\sin^2\theta}{2\cos\theta} = \frac{1}{2}\sin\theta\tan\theta \approx \frac{\theta^2}{2}$$

7. (i) $u = \frac{dy}{dx} + g(x)y$

Thus

$$\frac{du}{dx} = \frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y$$

As $\frac{du}{dx} + f(x)u = h(x)$

$$\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + g'(x)y + f(x)\left(\frac{dy}{dx} + g(x)y\right) = h(x)$$

that is

$$\frac{d^2y}{dx^2} + \big(g(x) + f(x)\big)\frac{dy}{dx} + \big(g'(x) + f(x)g(x)\big)y = h(x)$$

as required.

(ii)

$$g(x) + f(x) = 1 + \frac{4}{x}$$

and so $f(x) = 1 + \frac{4}{x} - g(x)$

$$g'(x) + f(x)g(x) = \frac{2}{x} + \frac{2}{x^2}$$

so

$$g'(x) + \left(1 + \frac{4}{x} - g(x)\right)g(x) = \frac{2}{x} + \frac{2}{x^2}$$

as requested.

If $g(x) = kx^n$, $g'(x) = knx^{n-1}$

$$knx^{n-1} + \left(1 + \frac{4}{x} - kx^n\right)kx^n = \frac{2}{x} + \frac{2}{x^2}$$

$$-k^2x^{2n+2} + kx^{n+2} + k(n+4)x^{n+1} - 2x - 2 = 0$$

Considering the x^{2n+2} term,

either it is eliminated by the x^{n+2} term, in which case, $2n + 2 = n + 2$ and $-k^2 + k = 0$

which would imply $n = 0$ and $k = 0$ or $k = 1$

$k = 0$ is not possible $(-2x - 2 = 0)$; $n = 0$, $k = 1$ would give $4x - 2x - 2 = 0$ so not possible

Or it is eliminated by the x^{n+1} term, in which case, $2n + 2 = n + 1$ which implies $n = -1$ and thus $-k^2 + k(n + 4) - 2 = 0$ and considering the other two terms $k - 2 = 0$

$k = 2$ and $n = -1$ satisfy $-k^2 + k(n + 4) - 2 = 0$ so these are possible values.

So $g(x) = \frac{2}{x}$ and as $f(x) = 1 + \frac{4}{x} - g(x)$, $f(x) = 1 + \frac{2}{x}$ $h(x) = 4x + 12$

$\frac{du}{dx} + f(x)u = h(x)$ is thus $\frac{du}{dx} + \left(1 + \frac{2}{x}\right)u = 4x + 12$

The integrating factor is

$$e^{\int\left(1+\frac{2}{x}\right)dx} = e^{x+2\ln x} = x^2e^x$$

Thus

$$x^2e^x\frac{du}{dx} + (x^2 + 2x)e^xu = (4x + 12)x^2e^x = (4x^3 + 12x^2)e^x$$

Integrating with respect to x

$$x^2e^xu = \int(4x^3 + 12x^2)e^x\ dx = 4x^3e^x + c$$

As $u = \frac{dy}{dx} + g(x)y$, and $g(x) = \frac{2}{x}$, when $x = 1$, $y = 5$, $\frac{dy}{dx} = -3$, we have $u = -3 + 2 \times 5$

That is $u = 7$, so $7e = 4e + c$, which means $c = 3e$

So $u = 4x + 3e\ \frac{e^{-x}}{x^2}$

$$\frac{dy}{dx} + \frac{2}{x}y = 4x + 3e\ \frac{e^{-x}}{x^2}$$

This has integrating factor

$$e^{\int\frac{2}{x}dx} = e^{2\ln x} = x^2$$

So

$$x^2\frac{dy}{dx} + 2xy = 4x^3 + 3e\ e^{-x}$$

Integrating with respect to x

$$x^2y = \int 4x^3 + 3e\ e^{-x}\ dx = x^4 - 3ee^{-x} + c'$$

when $x = 1$, $y = 5$ so $5 = 1 - 3 + c'$ which means $c' = 7$

Therefore,
$y = x^2 + \frac{7}{x^2} - \frac{3e^{-x+1}}{x^2}$

8. (i) All terms of the sequence are positive integers because they are all either equal to a previous term or the sum of two previous terms which are positive integers.

Thus, for $k \geq 1$, as $u_{2k} = u_k$ and $u_{2k+1} = u_k + u_{k+1}$, $u_{2k+1} - u_{2k} = u_{k+1} \geq 1$

Also, $u_{2k+1} - u_{2k+2} = u_k + u_{k+1} - u_{k+1} = u_k \geq 1$. Thus, the required result is proved for terms from the third onwards. (The only terms not included in this proof are the first two, which are in case both equal to 1).

(ii) Suppose that $u_{2k} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_k = c$, as $u_{2k} = u_k$, and $u_{k+1} = d - c \geq 1$ as $u_{2k+1} = u_k + u_{k+1}$ and using (i). Then as d and c share a common factor greater than one, d-c and c share a common factor greater than one. So, two earlier terms in the sequence do share the same common factor.

Likewise, suppose that $u_{2k+2} = c$, and that $u_{2k+1} = d$, for $k \geq 1$, where d and c share a common factor greater than one, then $u_{k+1} = c$ and $u_k = d - c$ giving the same result.

This is true for pairs of consecutive terms from the second term (and third) onwards. Repeating this argument, we find that it would imply that the first two terms would share a common factor greater than one, which is a contradiction. Hence any two consecutive terms are co-prime.

(iii) For $k \geq 1$, and $m \geq 1$ suppose that $u_{2k} = c$ and $u_{2k+1} = d$, and that $u_{2k+m} = c$ and $u_{2k+m+1} = d$, then as $d > c$, $2k + m$ is even, so m is even, say $2n$. Thus, $u_k = c$ and $u_{k+1} = d - c$, and $u_{k+n} = c$ and $u_{k+n+1} = d - c$. That is, an earlier pair of terms would appear consecutively.

Likewise, if $u_{2k+2} = c$ and $u_{2k+1} = d$, and that $u_{2k+m+2} = c$ and $u_{2k+m+1} = d$, the same argument applies.

So the argument can be repeated down to the first two terms, which are of course equal, and it would imply a later pair are likewise which contradicts (i).

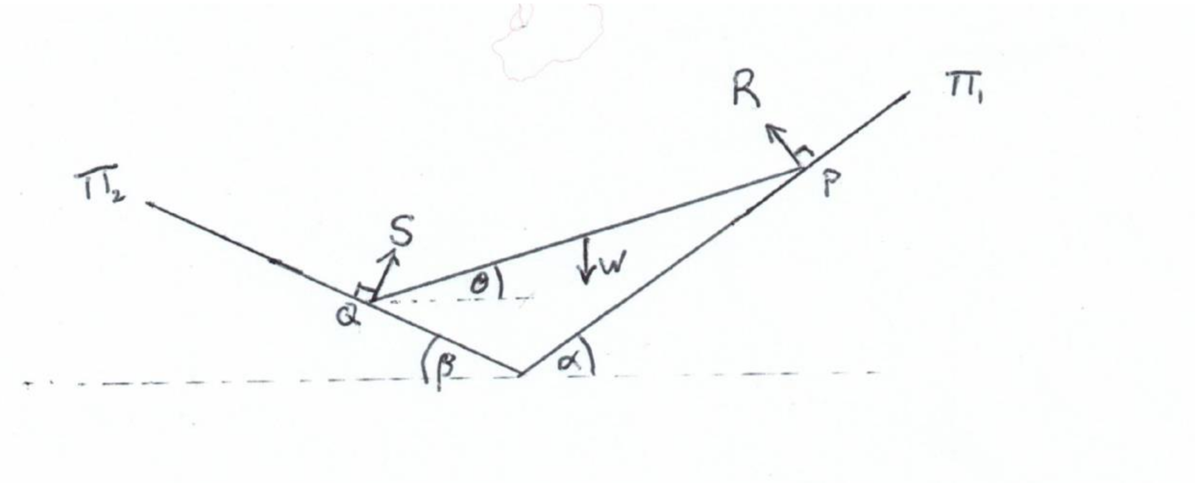
(iv) If (a, b) does not occur, where a and b are coprime and $a > b$, then there does not exist k such that $u_{2k+1} = a$ and $u_{2k+2} = b$. Therefore there cannot exist a *k* such that $u_{k+1} = b$ and $u_k = a - b$, the sum of which is a , which is smaller than $a + b$.

If (a, b) does not occur, where a and b are coprime and $a < b$, then there does not exist k such that $u_{2k} = a$ and $u_{2k+1} = b$. Therefore there cannot exist a *k* such that $u_k = a$ and $u_{k+1} = b - a$, the sum of which is b, which is smaller than $a + b$.

(v) Suppose that there exists an ordered pair of coprime integers (a,b) which does not occur consecutively in the sequence. Then by part (iv) the pair (a-b, b) [if a>b] or (a, b-a) [if b>a] (which has a smaller sum) does not occur. Repeating this means that a coprime pair with sum <3 does not occur. The only coprime pair of integers with sum <3 is (1, 1) which are the first two terms. Contradiction and so every ordered pair of coprime integers occurs in the sequence and by (iii) only occurs once. Therefore, there exists an n , and that n is unique such that

$q = \frac{u_n}{u_{n+1}}$, for any positive rational q (which is expressed in lowest form). So the inverse of f exists.

9. (i)



Resolving vertically, (1)

$$R\cos\alpha + S\cos\beta = W$$

Resolving horizontally, (2)

$$R\sin\alpha = S\sin\beta$$

Taking moments about Q, (3)

$$Wl\cos\theta = 2Rl\cos(\alpha - \theta)$$

Dividing (3) by l gives (4)

$$W\cos\theta = 2R\cos(\alpha - \theta)$$

Multiplying (1) by $\cos\theta\ \sin\beta$ gives

$$R\cos\alpha\ \cos\theta\ \sin\beta + S\cos\beta\cos\theta\ \sin\beta = W\cos\theta\ \sin\beta$$

Using (2) and (4) to substitute for $S\sin\beta$ and $W\cos\theta$ respectively,

$$R\cos\alpha\ \cos\theta\ \sin\beta + R\sin\alpha\cos\beta\cos\theta = 2R\cos(\alpha - \theta)\ \sin\beta$$

Thus

$$\cos\alpha\ \cos\theta\ \sin\beta + \sin\alpha\cos\beta\cos\theta = 2\cos\alpha\ \cos\theta\ \sin\beta + 2\sin\alpha\ \sin\theta\sin\beta$$

and so

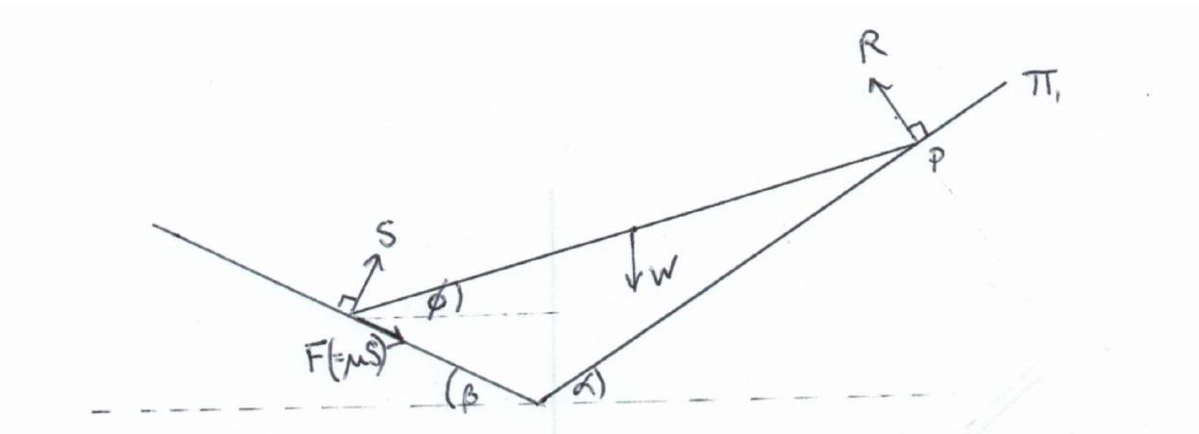
$$\sin\alpha\cos\beta\cos\theta - \cos\alpha\ \cos\theta\ \sin\beta = \ 2\sin\alpha\ \sin\theta\sin\beta$$

Dividing by $\sin\alpha\ \cos\theta\sin\beta$ gives

$$\cot\beta - \cot\alpha = 2\tan\theta$$

as required.

(ii)



Resolving vertically, (5)

$$R\cos\alpha + S\cos\beta = W + \mu S\sin\beta$$

Resolving horizontally, (6)

$$R\sin\alpha = S\sin\beta + \mu S\cos\beta$$

Taking moments about Q, and dividing by by l gives as before (4)

$$W\cos\phi = 2R\cos(\alpha - \phi)$$

Multiplying (5) by $(\sin\beta + \mu\cos\beta)\cos\phi$ gives

$$R\cos\alpha\,(\sin\beta + \mu\cos\beta)\cos\phi + S\cos\beta\,(\sin\beta + \mu\cos\beta)\cos\phi$$
$$= W(\sin\beta + \mu\cos\beta)\cos\phi + \mu S\sin\beta\,(\sin\beta + \mu\cos\beta)\cos\phi$$

Using (6) and (4) to substitute for $S(\sin\beta + \mu\cos\beta)$ and $W\cos\phi$ respectively,

$$R\cos\alpha\,(\sin\beta + \mu\cos\beta)\cos\phi + R\sin\alpha\cos\beta\cos\phi$$
$$= 2R\cos(\alpha - \phi)(\sin\beta + \mu\cos\beta) + \mu\, R\sin\alpha\sin\beta\cos\phi$$

Thus

$$\cos\alpha\,(\sin\beta + \mu\cos\beta)\cos\phi + \sin\alpha\cos\beta\cos\phi$$
$$= 2(\cos\alpha\cos\phi + \sin\alpha\ \sin\phi)(\sin\beta + \mu\cos\beta) + \mu\ \sin\alpha\sin\beta\cos\phi$$

So, dividing by $\sin\alpha\cos\beta\ \cos\phi$,

$$\cot\alpha\,(\tan\beta + \mu) + 1 = 2\cot\alpha\,(\tan\beta + \mu) + 2\tan\phi\ (\tan\beta + \mu) + \mu\tan\beta$$

$$1 = \cot\alpha\,(\tan\beta + \mu) + 2\tan\phi\ (\tan\beta + \mu) + \mu\tan\beta$$

From (i), $\cot\alpha = \cot\beta - 2\tan\theta$

so

$$1 = (\cot\beta - 2\tan\theta)(\tan\beta + \mu) + 2\tan\phi\ (\tan\beta + \mu) + \mu\tan\beta$$

$$1 - \mu\tan\beta - 1 - \mu\cot\beta = 2(\tan\phi - \tan\theta)(\tan\beta + \mu)$$

Hence,

$$2(\tan\theta - \tan\phi)(\tan\beta + \mu) = \mu\tan\beta + \mu\cot\beta = \mu(\tan\beta + \cot\beta)$$

$$\tan\beta + \cot\beta = \frac{\sin\beta}{\cos\beta} + \frac{\cos\beta}{\sin\beta} = \frac{\sin^2\beta + \cos^2\beta}{\sin\beta\ \cos\beta} = \frac{1}{\sin\beta\ \cos\beta}$$

and so,

$$\tan\theta - \tan\phi = \frac{\mu}{2\sin\beta\ \cos\beta(\tan\beta + \mu)}$$

That is

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta}$$

as required.

Alternative method using concurrency principle

10. If the extension in the equilibrium position is , then

$$mg = \frac{kmgd}{a}$$

Thus, $d = \frac{a}{k}$

If the extension when the particle is released is $d + x$, then the equation of motion is

$$m\ddot{x} = mg - \frac{kmg(d+x)}{a} = mg - \frac{kmgd}{a} - \frac{kmgx}{a} = -\frac{kmgx}{a}$$

$$\ddot{x} = -\frac{kgx}{a}$$

This is simple harmonic motion with period $\frac{2\pi}{\Omega}$ where $\Omega^2 = \frac{kg}{a}$, i.e. $kg = a\Omega^2$ as required.

Let y be the displacement of the platform below the centre point of its oscillation,

then, $y = b - x$ and $\ddot{y} = -\omega^2 y = -\omega^2(b - x)$ (x newly defined as in question)

Thus, the equation of motion of the particle becomes

$$m\,\ddot{y} = mg - R - \frac{kmg(h - a - x)}{a}$$

So,

$$-m\omega^2(b - x) = mg - R - \frac{ma\Omega^2(h - a - x)}{a}$$

That is,

$$R = mg + m\Omega^2(a + x - h) + m\omega^2(b - x)$$

as required.

To remain in contact, $R \geq 0$ for $0 \leq x \leq 2b$

$$R = mg + m\Omega^2(a - h) + m\omega^2 b + m(\Omega^2 - \omega^2)x$$

so if $\omega < \Omega$, the minimum value of R is $mg + m\Omega^2(a - h) + m\omega^2 b$, (when $x = 0$)

thus $mg + m\Omega^2(a - h) + m\omega^2 b \geq 0$

Rearranging,

$$h \leq \frac{g + \omega^2 b}{\Omega^2} + a = \frac{a}{k} + \frac{\omega^2 b}{\Omega^2} + a = a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}$$

as required.

If $\omega > \Omega$, minimum value of R is $mg + m\Omega^2(a - h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2)$ (at $x = 2b$)

Thus,

$$mg + m\Omega^2(a-h) + m\omega^2 b + 2mb(\Omega^2 - \omega^2) \geq 0$$

and so,

$$h \leq \frac{g+\omega^2 b}{\Omega^2} + a + \frac{2b(\Omega^2-\omega^2)}{\Omega^2} = a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} + 2b - \frac{2\omega^2 b}{\Omega^2} = a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b$$

Thus, if $\omega < \Omega$, $h \leq a\left(1+\frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2} < a\left(1+\frac{1}{k}\right) + b$;

if $\omega > \Omega$, $h \leq a\left(1+\frac{1}{k}\right) - \frac{\omega^2 b}{\Omega^2} + 2b < a\left(1+\frac{1}{k}\right) + b$

If $\omega = \Omega$, then $R = mg + m\Omega^2(a-h) + m\omega^2 b \geq 0$,

so,

$$h \leq a\left(1+\frac{1}{k}\right) + b$$

Alternative

stem measuring y below A

$$m\ddot{y} = mg - \frac{kmg(y-a)}{a}$$

$$\ddot{y} = -\frac{kg}{a}\,y + kg\left(1+\frac{1}{k}\right)$$

and for platform introduced

$$m\,\ddot{y} = mg - R - \frac{kmg(y-a)}{a}$$

11. (i)

$$P(Y \leq y) = P(f(X) \leq y)$$
$$= P\left(X \geq f^{-1}(y)\right)$$

as f is a strictly decreasing function

$$= P\left(X \geq f(y)\right)$$

$$= \frac{b - f(y)}{b - a}$$

because X is uniformly distributed on [a,b].

Thus, the pdf of Y is

$$\frac{d}{dy}\left(\frac{b - f(y)}{b - a}\right) = \frac{-f'(y)}{b - a}$$
$$y \in [a, b]$$

$$E(Y^2) = \int_a^b y^2 \frac{-f'(y)}{b - a}\, dy = \left[y^2 \frac{-f(y)}{b - a}\right]_a^b - \int_a^b 2y \frac{-f(y)}{b - a}\, dy$$

by integration by parts

$$= \frac{-ab^2 + a^2 b}{b - a} + \int_a^b 2x \frac{f(x)}{b - a}\, dx = \frac{ab(a - b)}{b - a} + \int_a^b 2x \frac{f(x)}{b - a}\, dx = -ab + \int_a^b 2x \frac{f(x)}{b - a}\, dx$$

as required.

(ii) Considering Z as a function of X, it satisfies the three conditions for the function f in part (i), as trivially by the definition of c the first is satisfied, considering the graph or the derivative the second is, and by symmetry, the third is.

$$\frac{1}{Z} + \frac{1}{X} = \frac{1}{c}$$

so

$$\frac{1}{Z} = \frac{1}{c} - \frac{1}{X} = \frac{X - c}{cX}$$

and therefore

$$Z = \frac{cX}{X - c}$$

Therefore,

$$E(Z) = \int_a^b \frac{cx}{x - c} \frac{1}{b - a}\, dx = \frac{c}{b - a} \int_a^b \frac{x - c}{x - c} + \frac{c}{x - c}\, dx = \frac{c}{b - a} \int_a^b 1 + \frac{c}{x - c}\, dx$$

$$= \frac{c}{b-a}\ [x + \mathrm{c}\ln(x-c)]_a^b$$

$$= c + \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

From (i),

$$E(Z^2) = -ab + \int_a^b \frac{cx}{x-c}\,\frac{2x}{b-a}\,dx$$

$$= -ab + \frac{2c}{b-a}\int_a^b \frac{x^2}{x-c}\,dx = -ab + \frac{2c}{b-a}\int_a^b \frac{x^2-xc}{x-c} + \frac{xc-c^2}{x-c} + \frac{c^2}{x-c}\,dx$$

$$= -ab + \frac{2c}{b-a}\left[\frac{x^2}{2} + cx + c^2\ln(x-c)\right]_a^b$$

$$= -ab + c(a+b) + 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

$$= -ab + ab + 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) = 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right)$$

Thus,

$$Var(Z) = 2c^2 + \frac{2c^3}{b-a}\ln\left(\frac{b-c}{a-c}\right) - \left(c + \frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2$$

$$= c^2 - \left(\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2$$

As $Var(Z) > 0$, (Z is not a constant)

$$c^2 - \left(\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)\right)^2 > 0$$

and so as c and $\frac{c^2}{b-a}\ln\left(\frac{b-c}{a-c}\right)$ are both positive,

$$c > \frac{c^2}{b-a} \ln\left(\frac{b-c}{a-c}\right)$$

and similarly,

$$\ln\left(\frac{b-c}{a-c}\right) < \frac{b-a}{c}$$

as required.

12. (i) $P(X = x) = q^{x-1}p$ and $P(Y = y) = q^{y-1}p$ for $x, y \geq 1$

$$P(S = s) = P(X + Y = s) = \sum_{x=1}^{s-1} P(X = x, Y = s - x) = \sum_{x=1}^{s-1} q^{x-1}pq^{s-x-1}p$$

$$= \sum_{x=1}^{s-1} q^{s-2}p^2 = (s-1)p^2q^{s-2}$$

for $s \geq 2$.

$$P(T = t) = P(X = t, Y \leq t) + P(Y = t, X \leq t) - P(X = t, Y = t)$$

$$= 2q^{t-1}p \sum_{y=1}^{t} q^{y-1}p \; - q^{t-1}pq^{t-1}p$$

$$= 2q^{t-1}p^2 \frac{1 - q^t}{1 - q} - q^{2t-2}\,p^2 = 2q^{t-1}p(1 - q^t) - q^{2t-2}\,p^2$$

$$= q^{t-1}p\,(2 - 2q^t - q^{t-1}p) = pq^{t-1}(2 - 2q^t - (1 - q)q^{t-1}) = pq^{t-1}(2 - q^{t-1} - q^t)$$

for $t \geq 1$ as required.

(ii)

$$P(U = u) = \sum_{x=1}^{\infty} P(X = x, Y = x + u) + \sum_{x=1}^{\infty} P(Y = x, X = x + u)$$

for $u \geq 1$

$$= 2\sum_{x=1}^{\infty} q^{x-1}p\,q^{x+u-1}p = 2p^2\,q^u \sum_{x=1}^{\infty} q^{2x-2} = 2p^2\,q^u \frac{1}{1 - q^2}$$

$$= 2p^2\,q^u \frac{1}{p(1+q)} = \frac{2p\,q^u}{(1+q)}$$

and

$$P(U = 0) = \sum_{x=1}^{\infty} P(X = x, Y = x) = \sum_{x=1}^{\infty} q^{x-1}p\,q^{x-1}p = p^2 \sum_{x=1}^{\infty} q^{2x-2} = \frac{p^2}{1 - q^2} = \frac{p}{1+q}$$

$$P(W=w)=P(X=w,Y\geq w)+P(Y=w,X\geq w)-P(X=w,Y=w)$$

for $w\geq 1$

$$=2\sum_{y=0}^{\infty}q^{w-1}p\,q^{w+y-1}p-q^{w-1}p\,q^{w-1}p=2p^2q^{2w-2}\sum_{y-0}^{\infty}q^y-p^2q^{2w-2}$$

$$=\ p^2q^{2w-2}\ \frac{2}{1-q}-p^2q^{2w-2}=p^2q^{2w-2}\left(\frac{2}{1-q}-1\right)=p^2q^{2w-2}\frac{1+q}{1-q}$$

$$=pq^{2w-2}(1+q)$$

(iii) $S=2\ \Rightarrow X=1\,,Y=1$ and $T=3\ \Rightarrow X=3$ or $Y=3$ or both

Thus,

$P(S=2\,,T=3)=0$

However,

$P(S=2)=p^2\neq 0$ and $P(T=3)=pq^2(2-q^2-q^3)=pq^2(1-q^2+1-q^3)$

$$=pq^2\big(p(1+q)+p(1+q+q^2)\big)$$

$$=p^2q^2(2+2q+q^2)\neq 0$$

and so, $P(S=2\,,T=3)\ \neq\ P(S=2)\ \times\ P(T=3)$

(iv)

$U=u$ and $W=w\ \Rightarrow\ X=w,Y=w+u$ or $Y=w\,,X=w+u$ for $u>0$

So $P(U=u\,,W=w)=2q^{w-1}pq^{w+u-1}p=2p^2q^{2w+u-2}$

$$P(U=u)=\frac{2p\,q^u}{(1+q)}$$

and

$$P(W=w)=pq^{2w-2}(1+q)$$

so

$$P(U=u)\times P(W=w)=\frac{2p\,q^u}{(1+q)}\times pq^{2w-2}(1+q)=2p^2q^{2w+u-2}=P(U=u\,,W=w)$$

In the case $u=0$,

$U=0$ and $W=w\ \Rightarrow\ X=w,Y=w$

so $P(U=0\,,W=w)=q^{w-1}p\quad q^{w-1}p\ \ =p^2q^{2w-2}$

whereas $P(U=0)=\frac{p}{1+q}$ and $P(W=w)=pq^{2w-2}(1+q)$

so

$$P(U=0) \times P(W=w) = \frac{p}{1+q} \times pq^{2w-2}(1+q) = p^2 q^{2w-2} = P(U=0\,, W=w)$$

Thus, U and W are independent.

Pairing S and U – consider $S = 2$, $U = 3$

$S = 2 \Rightarrow X = 1\,, Y = 1$ which would imply $U = 0$

Thus $P(S=2) = p^2 \neq 0$ and $P(U=3) = \frac{2p\,q^3}{(1+q)} \neq 0$ whereas $P(S=2\,, U=3) = 0$ so

$P(S=2\,, U=3) \neq P(S=2) \times P(U=3)$ and S and U are not independent.

Pairing S and W - consider $S = 2$, $W = 3$

$S = 2 \Rightarrow X = 1\,, Y = 1$ which would imply $W = 1$

so $P(S=2\,, W=3) = 0$ whereas $P(S=2) \neq 0$ and $P(W=3) \neq 0$ so S and W are not independent.

Pairing T and U - consider $T = 1$, $U = 1$

$T = 1 \Rightarrow X = 1\,, Y = 1$ which would imply $U = 0$

so $P(T=1\,, U=1) = 0$ whereas $P(T=1) \neq 0$ and $P(U=1) \neq 0$ so T and U are not independent.

Pairing T and W - consider $T = 1$, $W = 2$

$T = 1 \Rightarrow X = 1\,, Y = 1$ which would imply $W = 1$

so $P(T=1\,, W=2) = 0$ whereas $P(T=1) \neq 0$ and $P(W=2) \neq 0$ so T and W are not independent.

Alternative (i)

$$P(T=t) = P(X=t, Y<t) + P(Y=t, X<t) + P(X=t, Y=t)$$

[or $P(T=t) = P(X=t, Y<t) + P(Y=t, X \leq t)$]

$$= 2q^{t-1}p \sum_{y=1}^{t-1} q^{y-1}p \; + q^{t-1}pq^{t-1}p$$

$$= 2q^{t-1}p^2 \,\frac{1-q^{t-1}}{1-q} + q^{2t-2}\,p^2 = 2q^{t-1}p(1-q^{t-1}) + q^{2t-2}\,p^2$$

$$= q^{t-1}p\,(2 - 2q^{t-1} + q^{t-1}p) = pq^{t-1}(2 - 2q^{t-1} + (1-q)q^{t-1}) = \; pq^{t-1}(2 - q^{t-1} - q^t)$$

for $t \geq 1$ as required.

(ii)

$$P(W = w) = P(X = w, Y > w) + P(Y = w, X > w) + P(X = w, Y = w)$$

for $w \geq 1$

$$= 2\sum_{y=1}^{\infty} q^{w-1}p\, q^{w+y-1}p + q^{w-1}p\, q^{w-1}p = 2p^2q^{2w-2}\sum_{y-1}^{\infty} q^y + p^2q^{2w-2}$$

$$= \; p^2q^{2w-1}\,\frac{2}{1-q} + p^2q^{2w-2} = p^2q^{2w-2}\left(\frac{2q}{1-q} + 1\right) = p^2q^{2w-2}\frac{1+q}{1-q}$$

$$= pq^{2w-2}(1+q)$$

Multiple alternatives for counter-examples for (iv)